ELEN E4810: Digital Signal Processing
Topic 6: Filters - Introduction

1. Simple Filters
2. Ideal Filters
3. Linear Phase and FIR filter types

1. Simple Filters

- Filter = system for altering signal in some ‘useful’ way
- LSI systems:
  - are characterized by $H(z)$ (or $h[n]$)
  - have different gains (& phase shifts) at different frequencies
  - can be designed systematically for specific filtering tasks

FIR & IIR

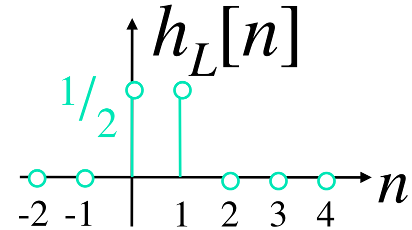
- FIR = finite impulse response
  ⇔ no feedback in block diagram

  ⇔ no poles (only zeros)
- IIR = infinite impulse response
  ⇔ feedback in block diagram

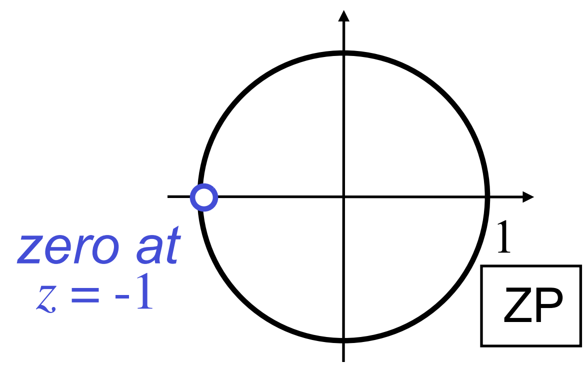
  ⇔ poles (and often zeros)

Simple FIR Lowpass

- $h_L[n] = \{\underset{\uparrow}{^1/_2}\ {^1/_2}\}$
  (2 pt moving avg.)



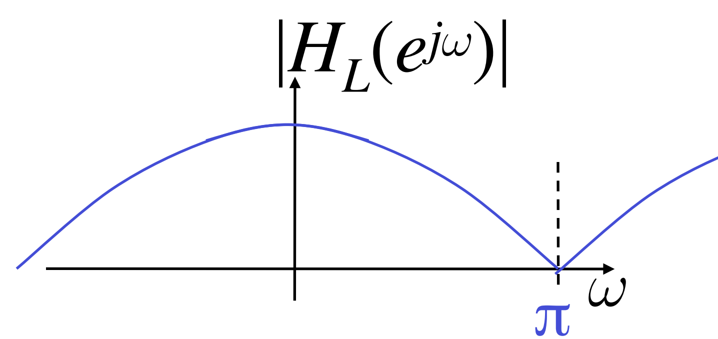
$$H_L(z) = \tfrac{1}{2}\left(1 + z^{-1}\right) = \frac{z+1}{2z}$$



$e^{j\omega/2} + e^{-j\omega/2}$

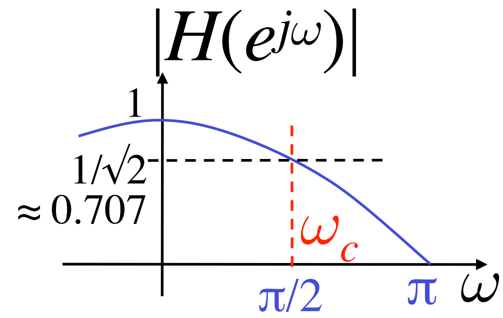
$$\Rightarrow H_L\left(e^{j\omega}\right) = e^{-j\omega/2}\cos(\omega/2)$$

$^1/_2$ sample delay



Simple FIR Lowpass

- Filters are often characterized by their cutoff frequency ω_c:



- Cutoff frequency is most often defined as the half-power point, i.e.

$$\left|H\left(e^{j\omega_c}\right)\right|^2 = \tfrac{1}{2}\max\left\{\left|H\left(e^{j\omega}\right)\right|^2\right\} \Rightarrow H = \tfrac{1}{\sqrt{2}} H_{\max}$$

- If $\left|H\left(e^{j\omega}\right)\right| = \cos(\omega/2)$

  then $\omega_c = 2\cos^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{2}$

deciBels

- Filter magnitude responses are often described in deciBels (dB)
- dB is simply a scaled log value:

$$dB = 20\log_{10}(level) = 10\log_{10}(power)$$

power = level²

- Half-power also known as 3dB point:

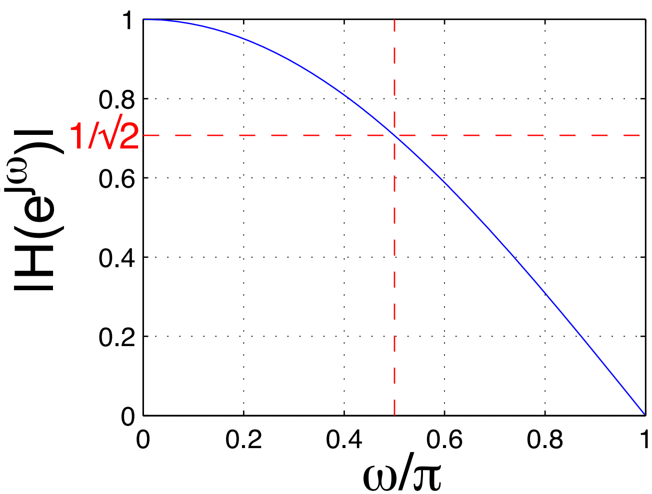
$$|H|_{cutoff} = \tfrac{1}{\sqrt{2}}|H|_{max}$$

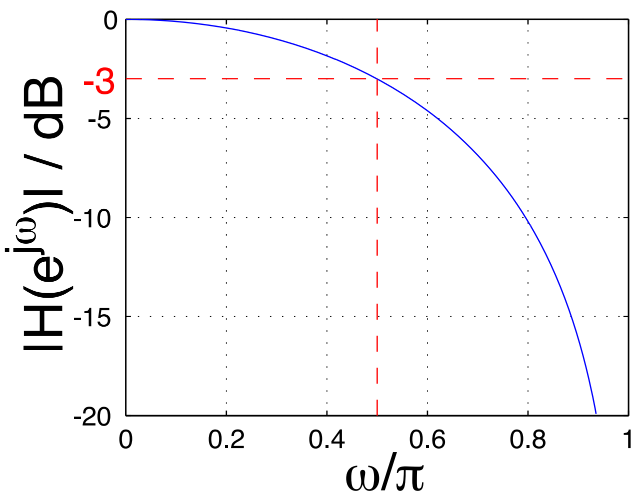
$$dB\{|H|_{cutoff}\} = dB\{|H|_{max}\} + 20\log_{10}\left(\tfrac{1}{\sqrt{2}}\right)$$

$$= dB\{|H|_{max}\} - 3.01$$

deciBels

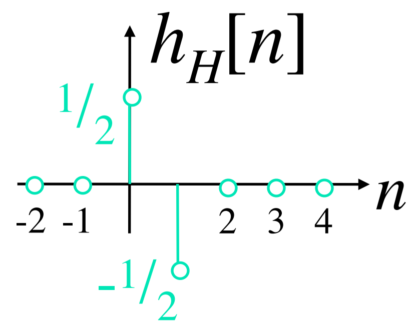
- We usually plot magnitudes in dB:



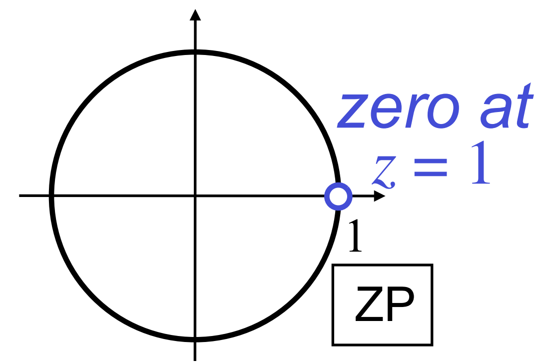


- A gain of 0 corresponds to $-\infty$ dB

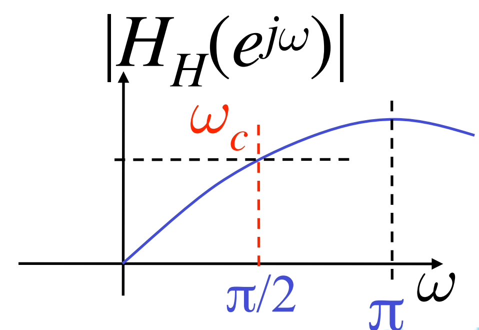
Simple FIR Highpass



- $h_H[n] = \{^1/_2\ -^1/_2\}$



$$H_H(z) = \tfrac{1}{2}\left(1 - z^{-1}\right) = \frac{z-1}{2z}$$



$$\Rightarrow H_H\left(e^{j\omega}\right) = je^{-j\omega/2}\sin(\omega/2)$$

- 3dB point $\omega_c = {}^{\pi}/_2$ (again)

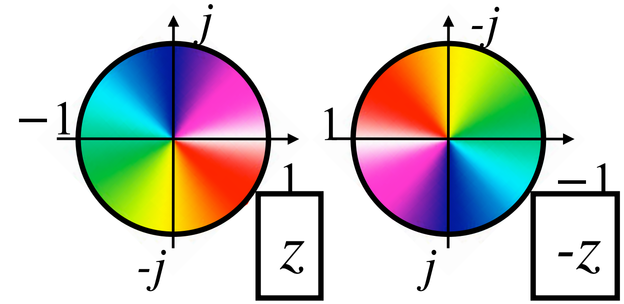
FIR Lowpass and Highpass

- Note:

  $h_L[n] = \{{}^1/_2\ {}^1/_2\}$ $\qquad$ $h_H[n] = \{{}^1/_2\ {-}{}^1/_2\}$

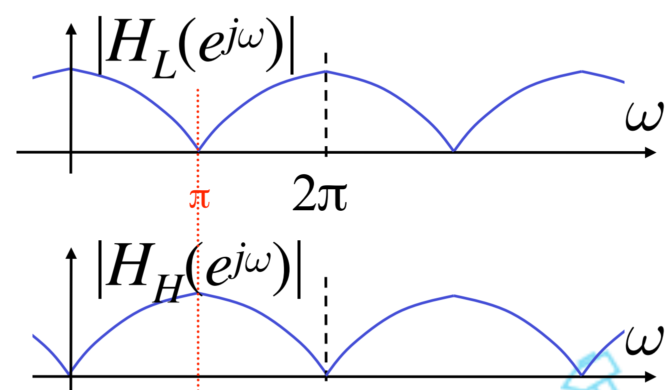
- i.e. $h_H[n] = (-1)^n h_L[n]$

  $\Rightarrow H_H(z) = H_L(-z)$



- i.e. 180° rotation of the z-plane,

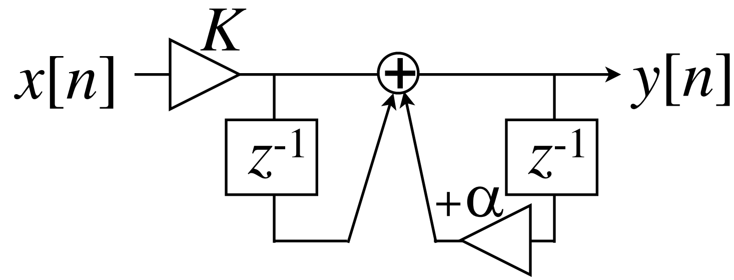
$\Rightarrow$ π shift of frequency response



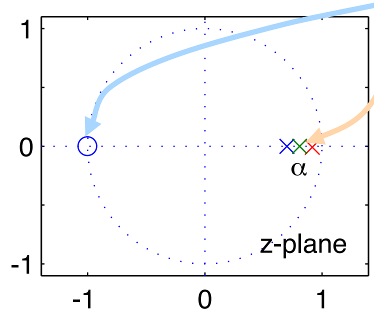
Simple IIR Lowpass

IIR → feedback, zeros and poles, conditional stability, $h[n]$ less useful

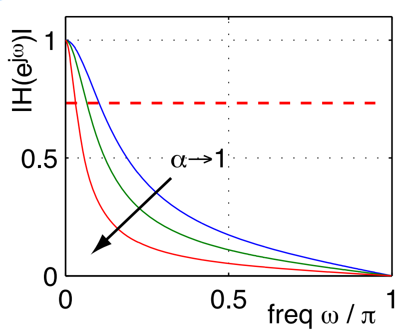
$$H_{LP}(z) = K \frac{1 + z^{-1}}{1 - \alpha z^{-1}}$$



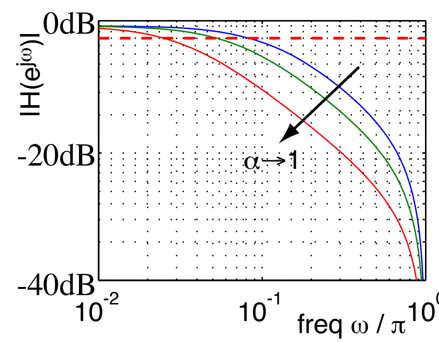
scale to make gain = 1 at ω = 0
→ $K = (1 - \alpha)/2$



pole-zero diagram



frequency response



FR on log-log axes

Simple IIR Lowpass

$$H_{LP}(z) = K\frac{1+z^{-1}}{1-\alpha z^{-1}}$$

max = 1
using $K=(1-\alpha)/2$

- Cutoff freq. ω_c from $\left|H_{LP}\left(e^{j\omega_c}\right)\right|^2 = \frac{\max}{2}$

$$\Rightarrow \frac{(1-\alpha)^2}{4}\frac{\left(1+e^{-j\omega_c}\right)\left(1+e^{j\omega_c}\right)}{\left(1-\alpha e^{-j\omega_c}\right)\left(1-\alpha e^{j\omega_c}\right)} = \frac{1}{2}$$

$$\Rightarrow \cos\omega_c = \frac{2\alpha}{1+\alpha^2} \Rightarrow \alpha = \frac{1-\sin\omega_c}{\cos\omega_c}$$

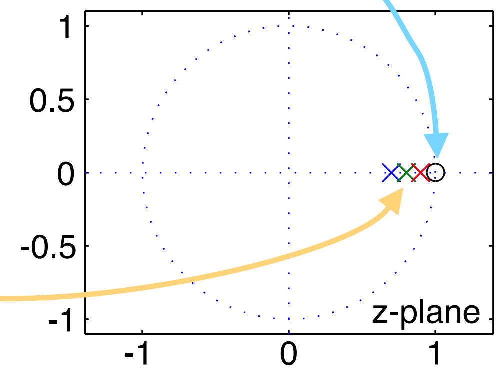
Design Equation

Simple IIR Highpass

$$H_{HP}(z) = K\frac{1 - z^{-1}}{1 - \alpha z^{-1}}$$

Pass $\omega = \pi \rightarrow H_{HP}(-1) = 1$

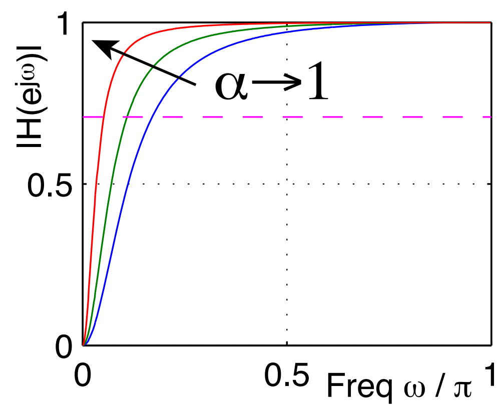
$\rightarrow K = (1+\alpha)/2$

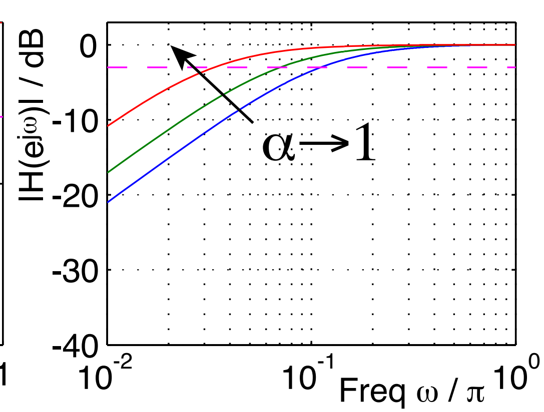


Design Equation:

$$\alpha = \frac{1 - \sin\omega_c}{\cos\omega_c}$$

(again)





Highpass and Lowpass

- Consider lowpass filter:

$$H_{LP}(e^{j\omega}) = \begin{cases} 1 & \omega \approx 0 \\ \sim 0 & \text{large } \omega \end{cases}$$

- Then:

$$\underbrace{1 - H_{LP}(e^{j\omega})} = \begin{cases} 0 & \omega \approx 0 \\ \sim 1 & \text{large } \omega \end{cases}$$

just another z poly

- *Highpass*
- *c/w* $(-1)^n h[n]$

- However, $|1 - H_{LP}(z)| \neq 1 - |H_{LP}(z)|$ (unless $H(e^{j\omega})$ is pure real - not for IIR)

Simple IIR Bandpass

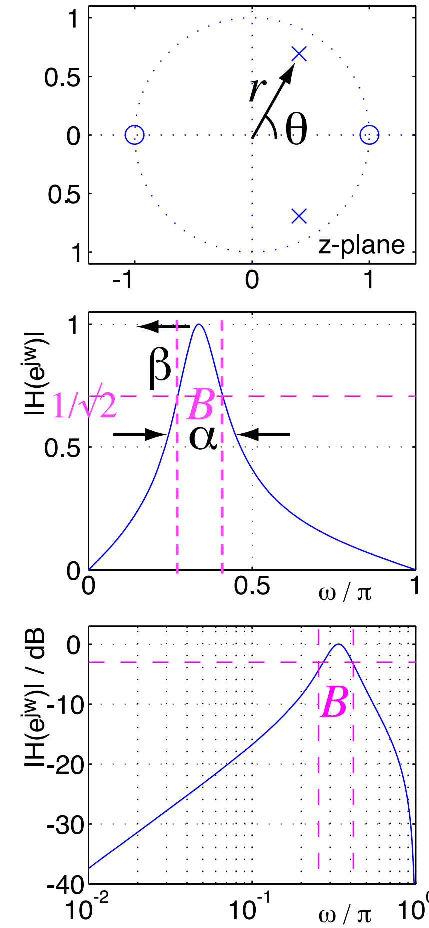
$$H_{BP}(z) = \frac{1-\alpha}{2}\,\frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}$$

$$= K\,\frac{\left(1+z^{-1}\right)\left(1-z^{-1}\right)}{1-2r\cos\theta\cdot z^{-1}+r^2 z^{-2}}$$

where $r = \sqrt{\alpha}$ $\quad \cos\theta = \dfrac{\beta(1+\alpha)}{2\sqrt{\alpha}}$

Design:

- Center freq $\omega_c = \cos^{-1}\beta$
- 3dB bandwidth $B = \cos^{-1}\left(\dfrac{2\alpha}{1+\alpha^2}\right)$



Simple Filter Example

- Design a second-order IIR bandpass filter with $\omega_c = 0.4\pi$, 3dB b/w of 0.1π

$$\omega_c = 0.4\pi \Rightarrow \beta = \cos\omega_c = 0.3090$$

$$B = 0.1\pi \Rightarrow \frac{2\alpha}{1+\alpha^2} = \cos(0.1\pi) \Rightarrow \alpha = 0.7265$$

$$\Rightarrow H_{BP}(z) = \frac{1-\alpha}{2}\frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}$$

$$= \frac{0.1367\left(1-z^{-2}\right)}{1-0.5335z^{-1}+0.7265z^{-2}}$$

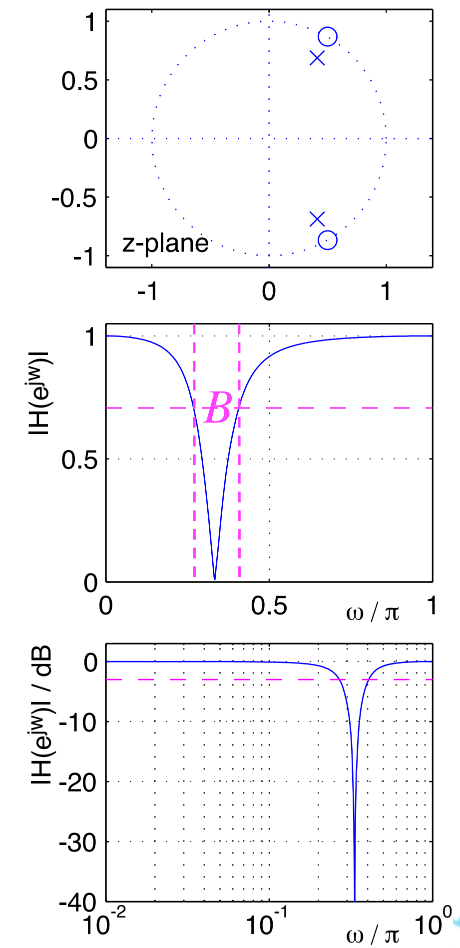
sensitive..

Simple IIR Bandstop

zeros at ω_c (per $1 - 2r\cos\theta\, z^{-1} + r^2 z^{-2}$)

$$H_{BS}(z) = \frac{1+\alpha}{2}\,\frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}}$$

same poles as H_{BP}



- Design eqns:

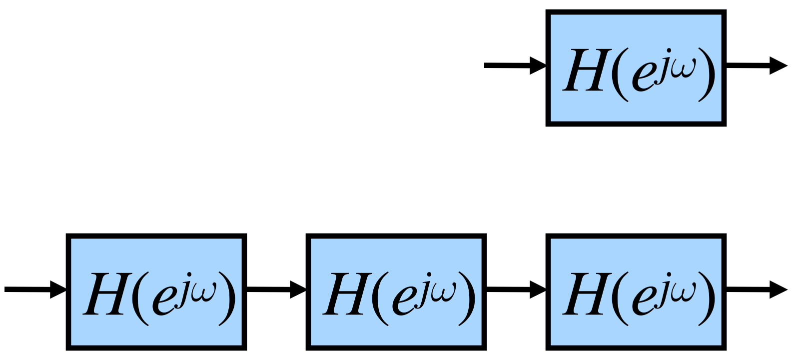
$$\omega_c = \cos^{-1}\beta \Rightarrow \beta = \cos\omega_c$$

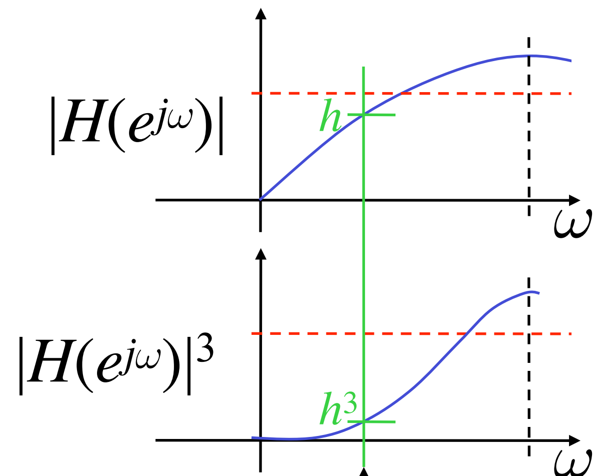
$$B = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)$$

$$\Rightarrow \alpha = \frac{1}{\cos B} - \sqrt{\frac{1}{\cos^2 B} - 1}$$

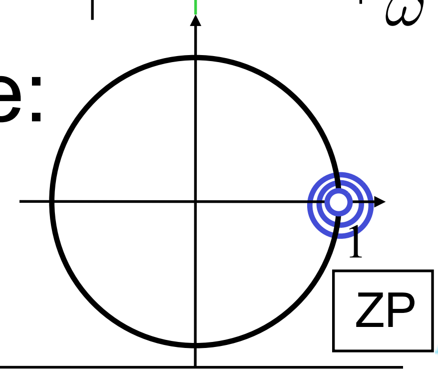
Cascading Filters

- Repeating a filter (cascade connection) makes its characteristics more abrupt:



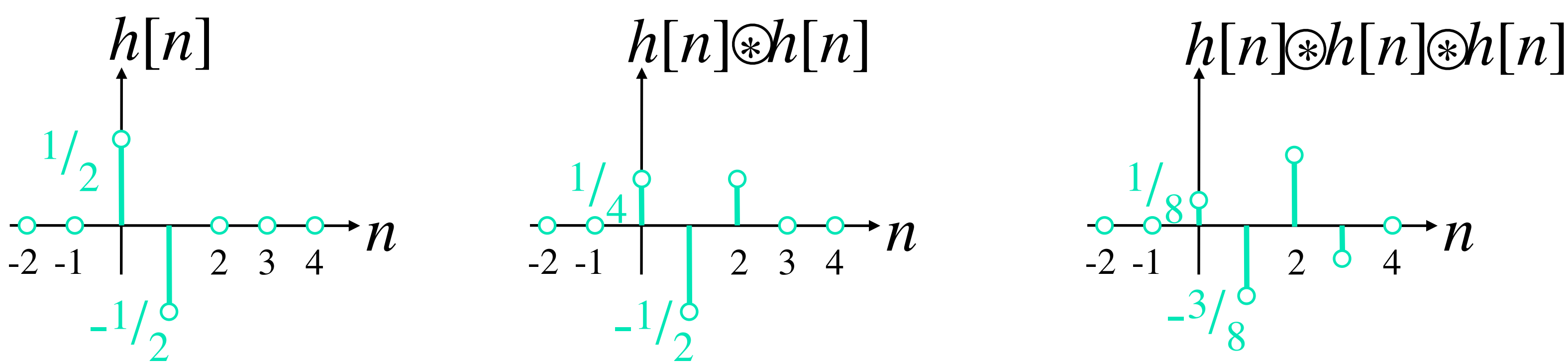


- Repeated roots in z-plane:



Cascading Filters

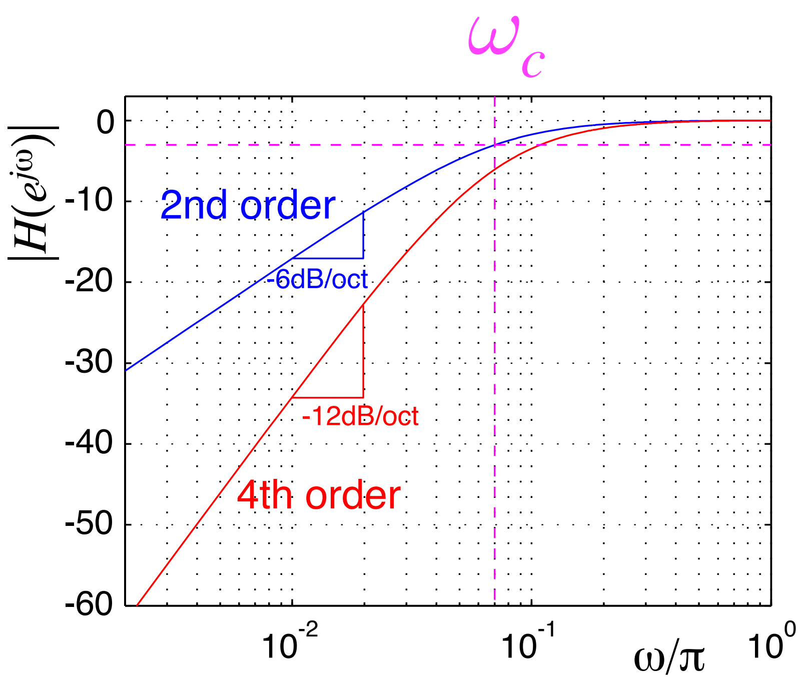
- Cascade systems are higher order e.g. longer (finite) impulse response:



- In general, cascade filters will not be optimal (...) for a given order

Cascading Filters

- Cascading filters improves rolloff slope:

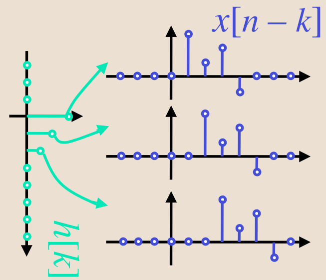


- But: 3dB cutoff frequency will change (gain at $\omega_c \rightarrow 3N$ dB)

Interlude: The Big Picture

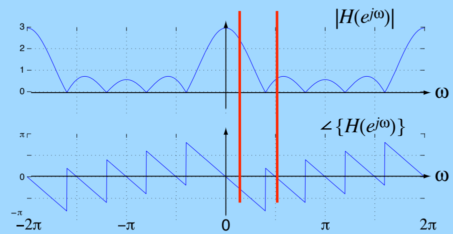
IR

$$y[n] = h[n] \circledast x[n]$$



DTFT

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$



DTFT

$$X(e^{j\omega}) = \sum_n x[n]e^{-j\omega n}$$

IDTFT

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

ZT

$$X(z) = \sum_n x[n]z^{-n}$$

IZT

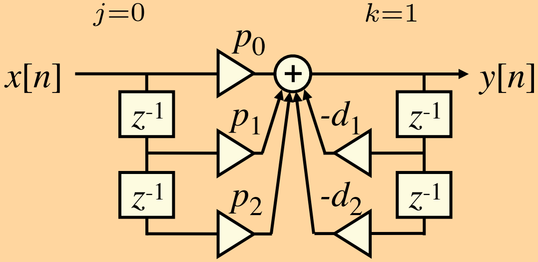
$$\lambda^n \mu[n] \leftrightarrow \frac{1}{1-\lambda z^{-1}} \quad |z| > |\lambda|$$

$$y_c[n] + y_p[n] = \sum_i \alpha_i \lambda_i^n + \beta\lambda_0^n \quad n \geq 0$$

$$X(e^{j\omega}) = X(z)|_{z=e^{j\omega}}$$

LCCDE

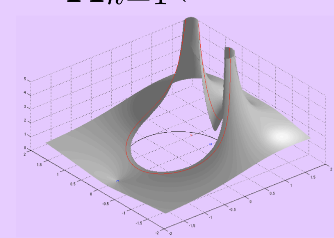
$$y[n] = \sum_{j=0}^{M} p_j x[n-j] - \sum_{k=1}^{N} d_k y[n-k]$$



$$\sum_j p_j x[n-j] \leftrightarrow \sum_j p_j z^{-j} X(z)$$

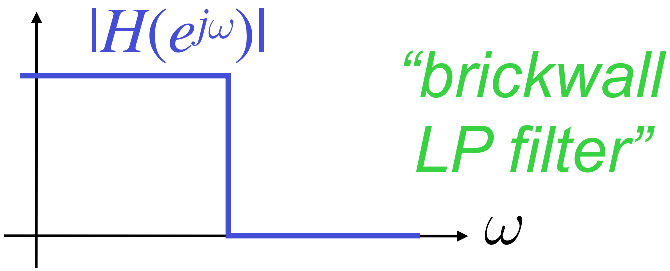
ZT

$$Y(z) = G\frac{\prod_{j=1}^{M}(1-\zeta_j z^{-1})}{\prod_{k=1}^{N}(1-\lambda_k z^{-1})}X(z)$$



2. Ideal filters

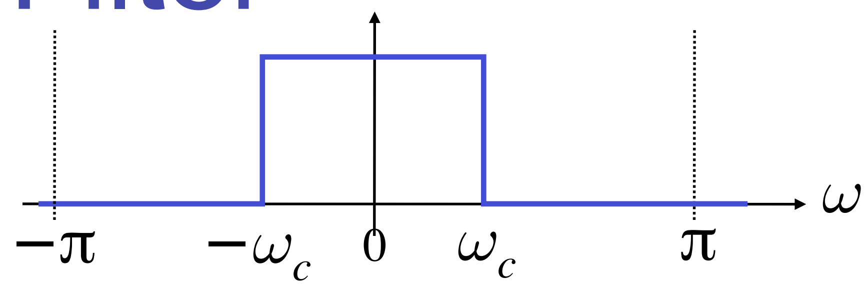
- Typical filter requirements:
  - gain = 1 for wanted parts (pass band)
  - gain = 0 for unwanted parts (stop band)
- "Ideal" characteristics would be like:
  - no phase distortion etc.



- What is this filter?
  - can calculate IR $h[n]$ as IDTFT of ideal response...

Ideal Lowpass Filter

- Given ideal $H(e^{j\omega})$:
  (assume $\theta(\omega) = 0$)



$$\Rightarrow \quad h[n] = IDTFT\left\{H\left(e^{j\omega}\right)\right\}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H\left(e^{j\omega}\right) e^{j\omega n}\, d\omega$$

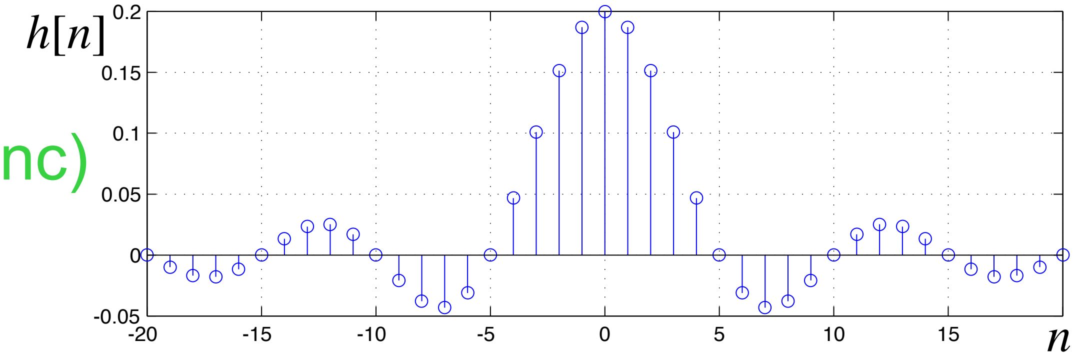
$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega$$

$$\Rightarrow h[n] = \frac{\sin \omega_c n}{\pi n}$$

Ideal lowpass filter

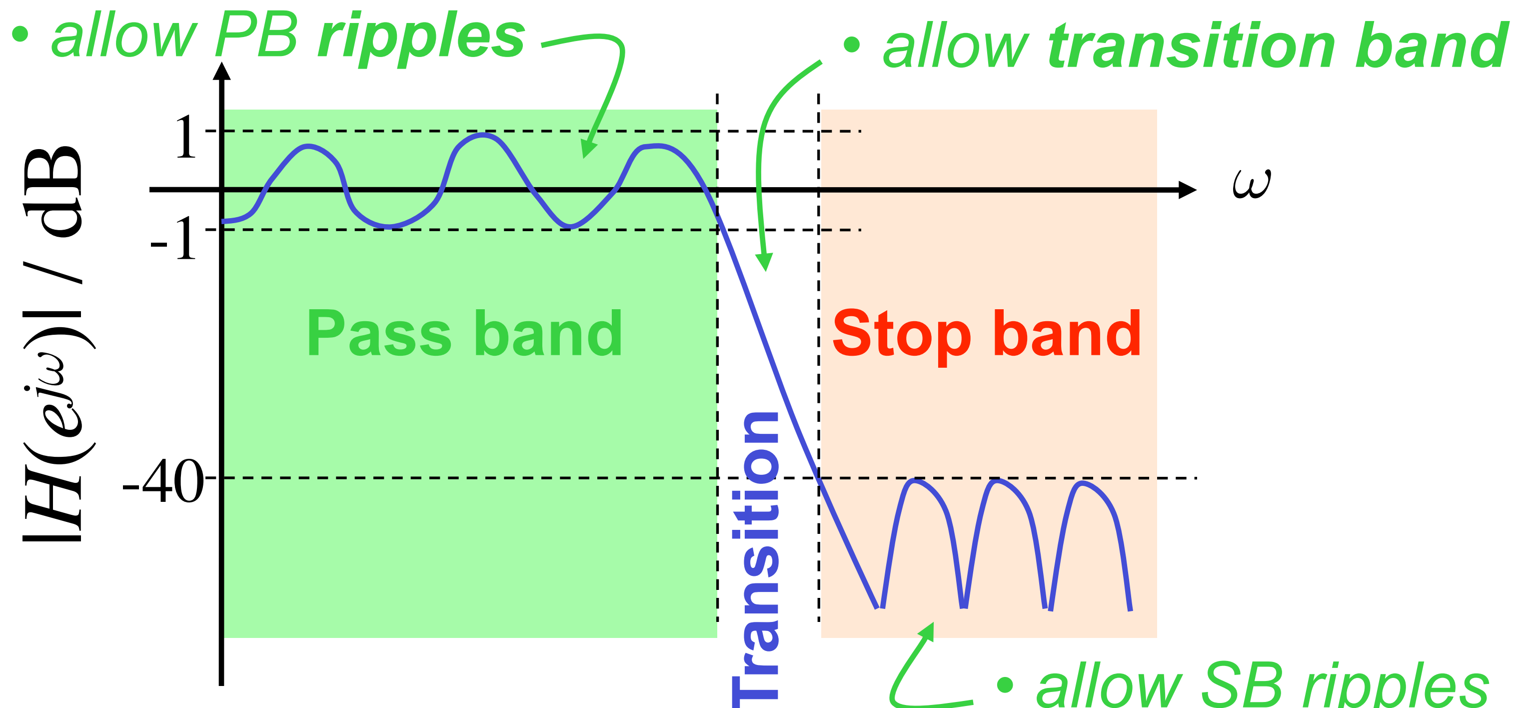
Ideal Lowpass Filter

$$h[n] = \frac{\sin \omega_c n}{\pi n}$$ (sinc)



- Problems!
  - doubly infinite ($n = -\infty .. \infty$)
  - no rational polynomial → very long FIR
  - excellent *frequency-domain* characteristics ↔ poor *time-domain* characteristics (blurring, ringing – a general problem)

Practical filter specifications



- lower-order realization (less computation)
- better time-domain properties (less ringing)
- easier to design...

3. Linear-phase Filters

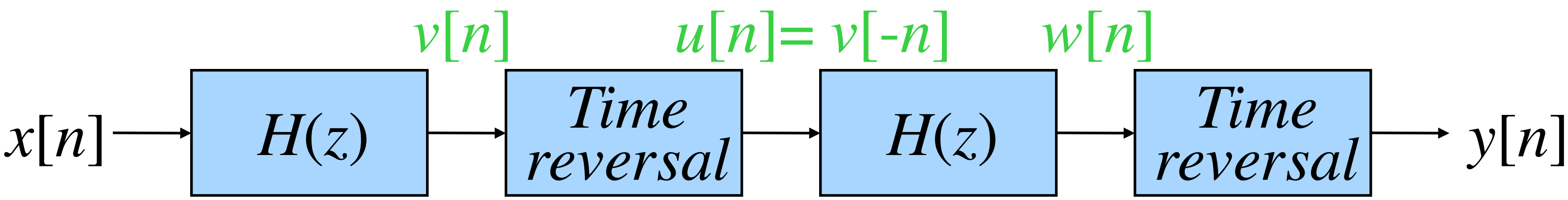
- $|H(e^{j\omega})|$ alone can hide *phase distortion*
  - differing delays for adjacent frequencies can mangle the signal
- Prefer filters with a flat phase response e.g. $\theta(\omega) = 0$ **"zero phase filter"**
- A filter with constant delay $\tau_p = D$ at all freqs has $\theta(\omega) = -D\omega$ **"linear phase"**

$$\Rightarrow H\left(e^{j\omega}\right) = e^{-jD\omega}\tilde{H}(\omega)$$

pure-real (zero-phase) portion

- Linear phase can 'shift' to zero phase

Time reversal filtering



- $v[n] = x[n] \circledast h[n] \;\rightarrow\; V(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$
- $u[n] = v[-n] \;\rightarrow\; U(e^{j\omega}) = V(e^{-j\omega}) = V^{*}(e^{j\omega})$ *if v real*
- $w[n] = u[n] \circledast h[n] \;\rightarrow\; W(e^{j\omega}) = H(e^{j\omega})U(e^{j\omega})$
- $y[n] = w[-n] \;\rightarrow\; Y(e^{j\omega}) = W^{*}(e^{j\omega})$

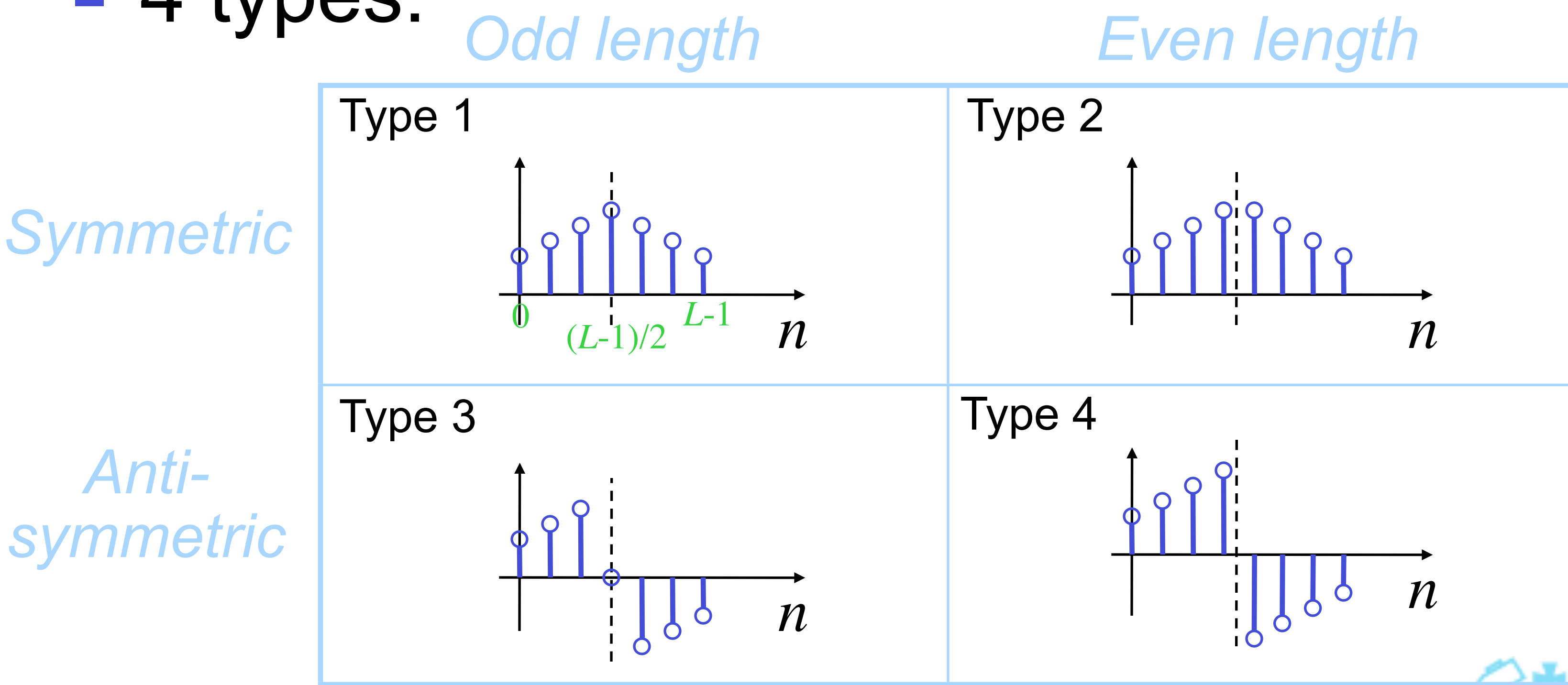
$$= (H(e^{j\omega})(H(e^{j\omega})X(e^{j\omega}))^{*})^{*}$$

$$\rightarrow\; Y(e^{j\omega}) = X(e^{j\omega})\left|H(e^{j\omega})\right|^{2}$$

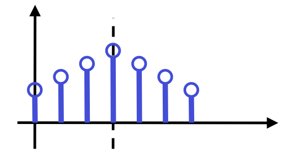
- Achieves zero-phase result
- Not causal! Need whole signal first

Linear Phase FIR filters

- (Anti)Symmetric FIR filters are almost the only way to get zero/linear phase
- 4 types:



Linear Phase FIR: Type 1

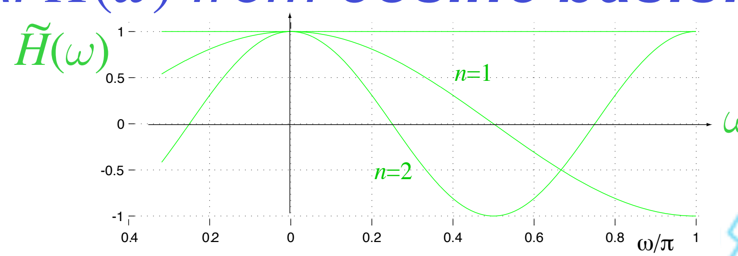


- *Length* L odd → *order* $N = L - 1$ even
- Symmetric → $h[n] = h[N - n]$
  ($h[N/2]$ unique)
- $$H\left(e^{j\omega}\right) = \sum_{n=0}^{N} h[n] e^{-j\omega n}$$
  $$= e^{-j\omega\frac{N}{2}}\left(h\left[\tfrac{N}{2}\right] + 2\sum_{n=1}^{N/2} h\left[\tfrac{N}{2} - n\right]\cos\omega n\right)$$

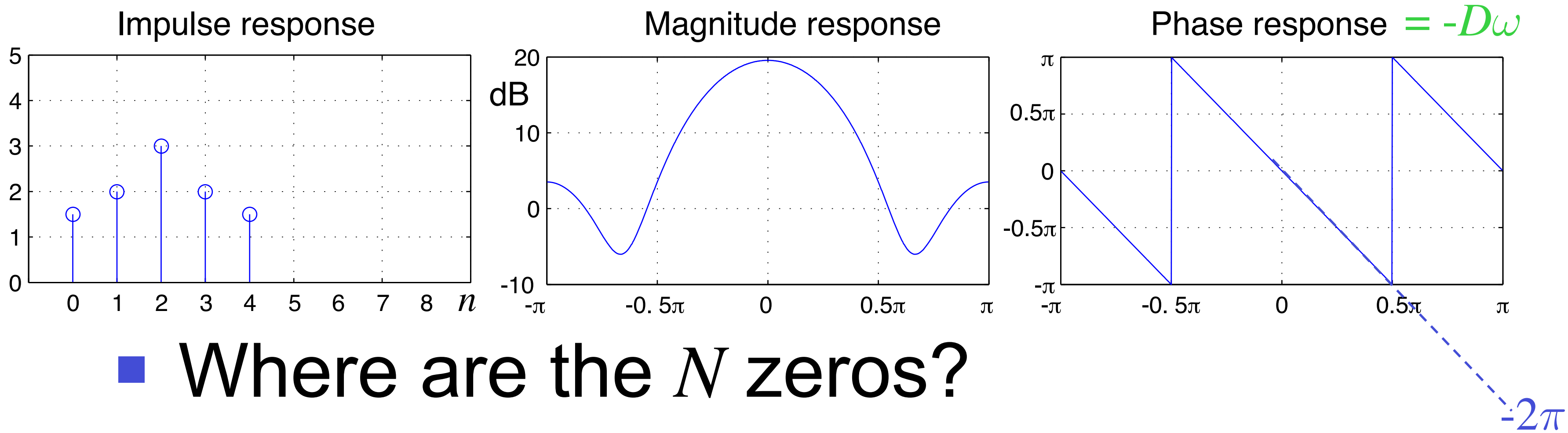
linear phase

$D = -\theta(\omega)/\omega = N/2$

pure-real $\tilde{H}(\omega)$ *from* ***cosine basis:***



Linear Phase FIR: Type 1



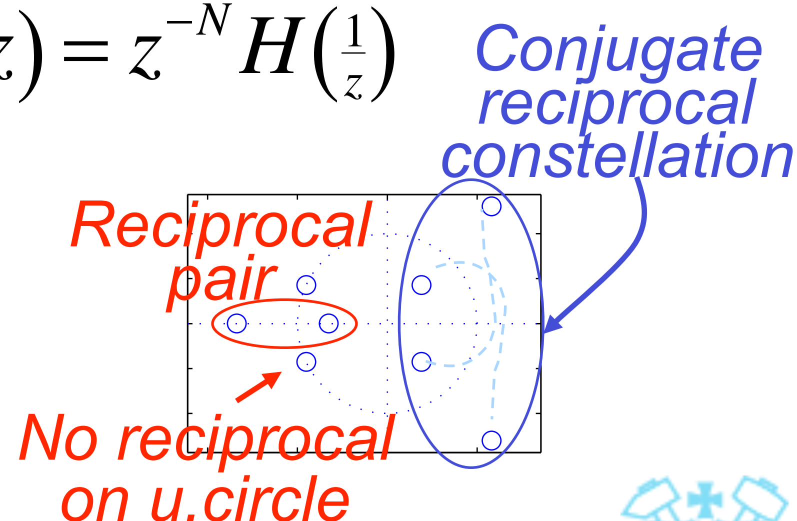
- Where are the N zeros?

$$h[n] = h[N-n] \Rightarrow H(z) = z^{-N} H\left(\tfrac{1}{z}\right)$$

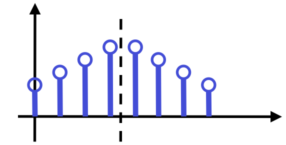
thus for a zero ζ

$$H(\zeta) = 0 \Rightarrow H\left(\tfrac{1}{\zeta}\right) = 0$$

Reciprocal zeros
(as well as cpx conj)



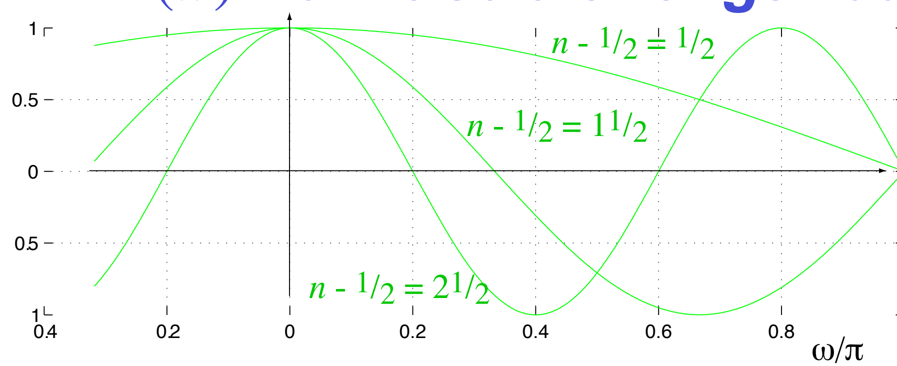
Linear Phase FIR: Type 2



- *Length L* even → *order $N = L - 1$* odd
- Symmetric → $h[n] = h[N - n]$
  (no unique point)
- $$H\left(e^{j\omega}\right) = e^{-j\omega\frac{N}{2}} \sum_{n=1}^{(N+1)/2} h\left[\tfrac{N+1}{2} - n\right] \cos\omega\left(n - \tfrac{1}{2}\right)$$

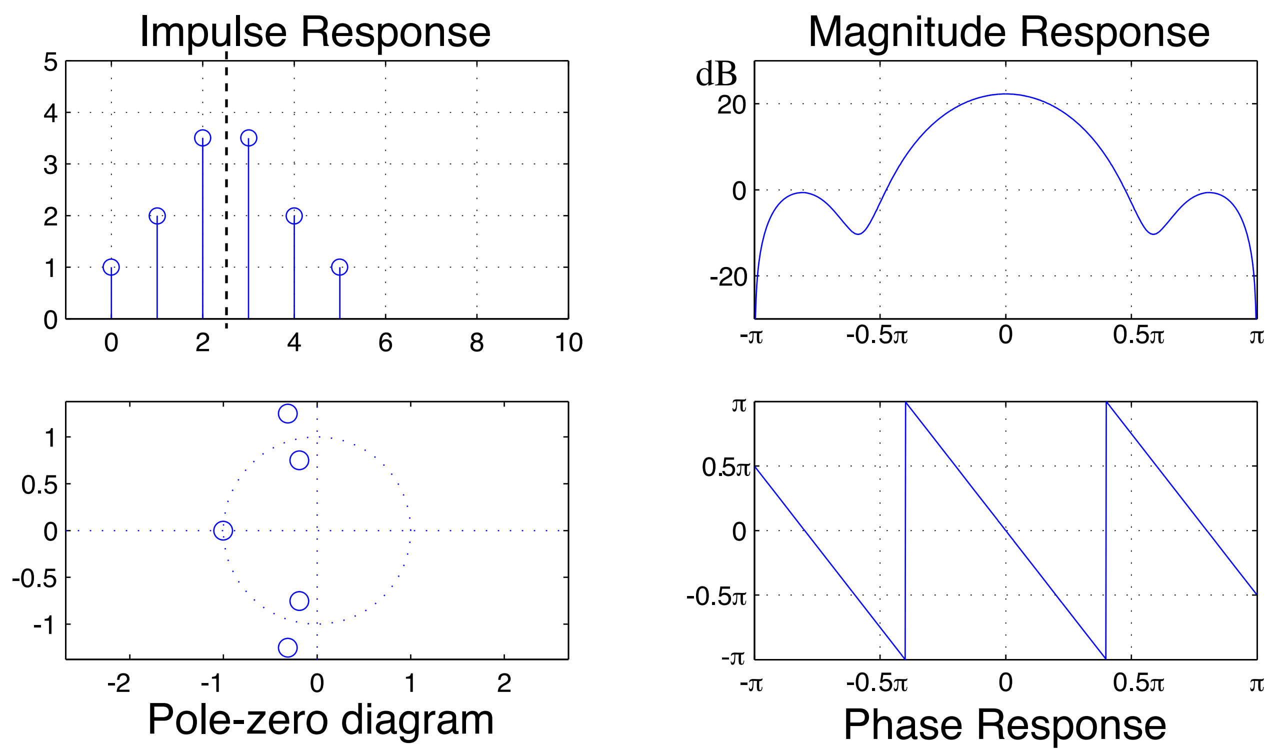
Non-integer delay of $N/2$ samples

*$\tilde{H}(\omega)$ from **double-length** cosine basis*



always zero at $\omega = \pi$

Linear Phase FIR: Type 2



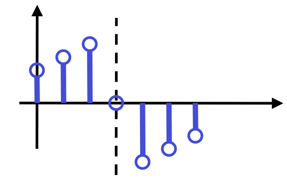
- Zeros: $H(z) = z^{-N} H\left(\frac{1}{z}\right)$

at z = -1, $H(-1) = (-1)^{N} H(-1) \Rightarrow H\left(e^{j\pi}\right) = 0$

LPF-like

odd

Linear Phase FIR: Type 3



- *Length* L odd → *order* $N = L - 1$ even
- Antisymmetric → $h[n] = -h[N - n]$

  $\Rightarrow h[N/2] = -h[N/2] = 0$

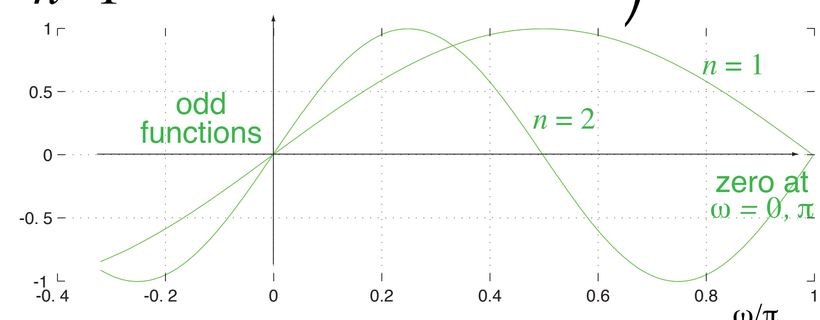
- $$H(e^{j\omega}) = \sum_{n=1}^{N/2} h[\tfrac{N}{2} - n]\left(e^{-j\omega(\frac{N}{2}-n)} - e^{-j\omega(\frac{N}{2}+n)}\right)$$

  $$= je^{-j\omega\frac{N}{2}}\left(2\sum_{n=1}^{N/2} h[\tfrac{N}{2} - n]\sin\omega n\right)$$

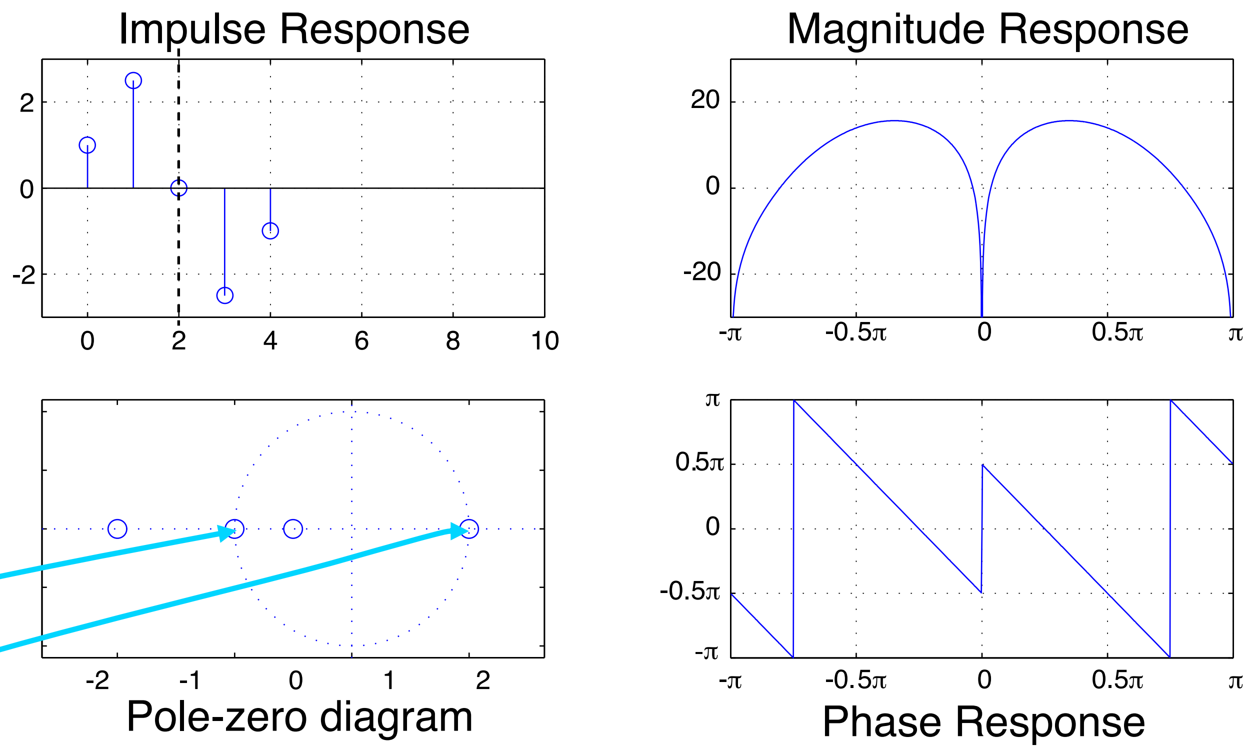
$\theta(\omega) = \pi/2 - \omega \cdot N/2$

Antisymmetric ⇒

$\pi/2$ phase shift in addition to linear phase



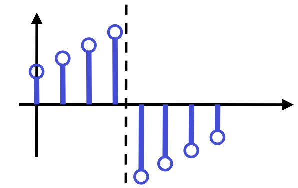
Linear Phase FIR: Type 3



- Zeros: $H(z) = -z^{-N} H\left(\frac{1}{z}\right)$

$$\Rightarrow H(1) = -H(1) = 0 \; ; \quad H(-1) = -H(-1) = 0$$

Linear Phase FIR: Type 4

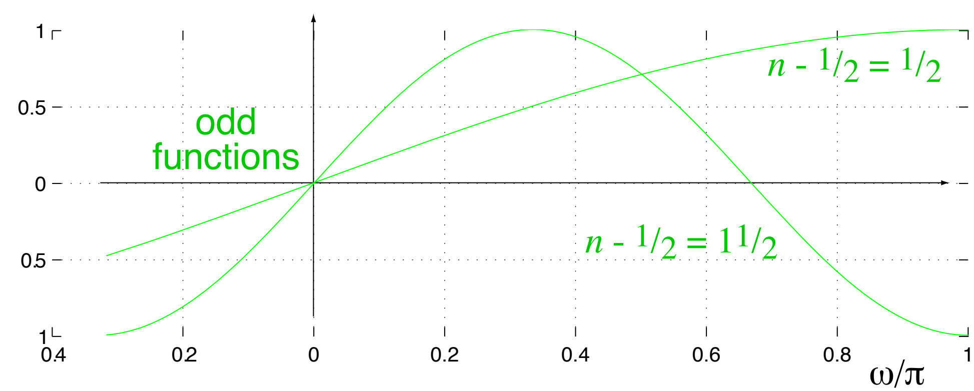


- *Length* L even → *order* $N = L - 1$ odd
- Antisymmetric → $h[n] = -h[N - n]$
  (no center point)
- $H(e^{j\omega}) = je^{-j\omega\frac{N}{2}} 2\sum_{n=1}^{N/2} h[\tfrac{N+1}{2} - n]\sin\omega(n - \tfrac{1}{2})$

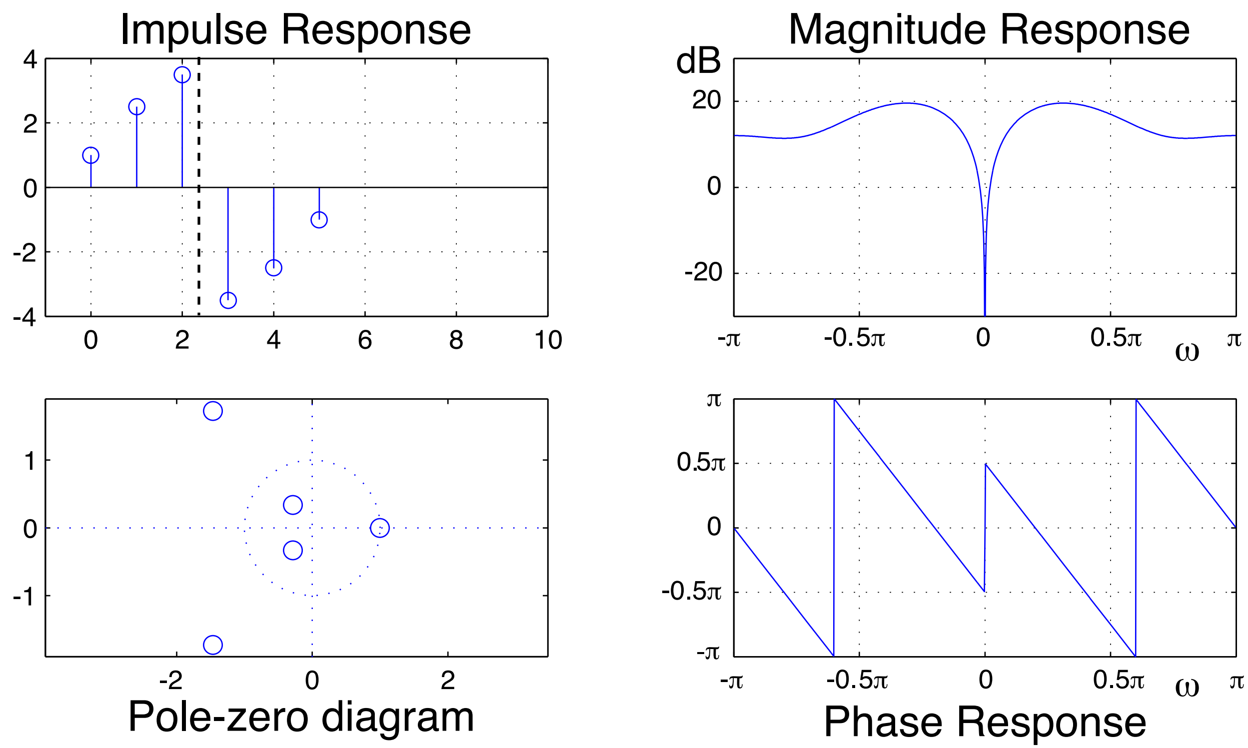
π/2 offset

fractional-sample delay

offset sine basis



Linear Phase FIR: Type 4



- Zeros: $H(1) = -H(1) = 0$

  (H(-1) OK because N is odd)

4 Linear Phase FIR Types